

# Potts model coupled to causal triangulations

*J. Cerda-Hernández*<sup>a</sup>

<sup>a</sup> Institute of Mathematics, Statistics and Scientific Computation,  
University of Campinas - UNICAMP,  
Rua Sérgio Buarque de Holanda 651, CEP 13083-859, Campinas, SP, Brazil.  
E-mail: javier@ime.usp.br.

## Abstract

In this work we study the annealed Potts model coupled to two dimensional causal triangulations with periodic boundary condition. Using duality on a torus, we provide a relation between the free energy of the Potts model coupled CTs and its dual. This duality relation follows from the FK representation for the Potts model. In order to determine a region where the critical curve for the model can be located we use the duality relation and the high-temperature expansion. This is done by outlining a region where the infinite-volume Gibbs measure exists and is unique and a region where the finite-volume Gibbs measure has no weak limit (in fact, does not exist if the volume is large enough). We also provide lower and upper bounds for the infinite-volume free energy.

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# 1 Introduction

Causal triangulation (CT), introduced by Ambjørn and Loll (see [11]), together with its predecessor a dynamical triangulation (DT), constitute attempts to provide a meaning to the formal expressions appearing in the path integral quantisation of gravity (see [1], [22] for an overview). The idea is to approximate the emerging geometries by CTs. As a result, we obtain a discrete version of the path integral where continuum geometries are replaced by a sum over all possible triangulations, where each triangulation is weighted by a Boltzmann factor  $e^{-\mu|T|}$ , with  $|T|$  standing for the size of the triangulation and  $\mu$  being the cosmological constant. Then, evaluation of the partition function is reduced to a purely combinatorial problem that can be solved utilizing the approach developed in the early work of Tutte [2, 3] or techniques based on random matrix models (see, e.g., [4]) and bijections to well-labelled trees (see [5, 6]).

Putting a spin system on the collection of all causal triangulations is generally interpreted as a coupling gravity with matter, which makes it interesting to study the  $q$ -state Potts model coupled to CTs from a physical point of view. The classical example of such model is the two-state Potts model (Ising model) coupled to a CT introduced in [14]. For the Ising model existence of Gibbs measures and phase transitions has been recently proved (see [14], [15], [20], [25] and [28] for details). For the numerical results for the 3-state Potts model coupled to CTs we refer the reader to [16]. In this work we focus on the  $q$ -state Potts model coupled to CTs for any  $q \geq 2$ . Our main goal is to derive properties of the phase transition for the model and the critical curve by defining a region in the quadrant of parameters  $\beta, \mu > 0$  where the infinite-volume free energy has a limit, which implies uniqueness of the Gibbs measure. In addition, we prove in Corollary 2 that the critical curve of the model is asymptotic if  $\beta$  is large to  $\frac{3}{2}\beta + \ln 2$ , and  $C(q) + o(\beta^2)$  if  $\beta$  is small, where  $C(q) = \ln 2\sqrt{q}$  for the Potts model coupled on CTs, and  $C(q) = \ln 2q$  for its dual. (see Figure 2). In order to obtain these results we utilize the FK-Potts models, introduced by Fortuin and Kasteleyn (see [21]). These representations were successfully employed in order to obtain important results for Ising and Potts models on the hypercubic lattice. Since this representation permits the use of geometric properties of the triangulations, we utilize it to derive a duality relation for the parameters of the model and asymptotic behavior of the critical curve.

In general, the FK-Potts model on a finite connected graph (not necessarily planar) is a model of edges of the graphs, where each edge is either

closed or open. The probability of a given configuration is proportional to

$$p^{\#\text{open edges}}(1-p)^{\#\text{closed edges}}q^{\#\text{clusters}},$$

where  $p \in [0, 1]$  and the cluster-weight  $q \in (0, \infty)$  are the parameters of the model. For  $q \geq 1$ , the model can be extended to an infinite graph. In this case the model exhibits a phase transition for some critical parameter  $p_c(q)$ , which depends on the geometry of the graph. In the case of planar graphs, there exists a relation between FK-Potts models on a graph and its dual, with the same parameter  $q$  and appropriately related parameters  $p$  and  $p^* = p^*(p)$ . For a detailed introduction to the FK-Potts model we refer the reader to [27]. In the case of FK-Potts model defined on a causal triangulation  $\mathbf{t}$  with periodic boundary condition (see Figure 1 for a geometric representation), the partition function of the FK-Potts model on  $\mathbf{t}$  cannot be written exactly as a partition function of a FK-Potts model on its dual  $\mathbf{t}^*$ , however it will be sufficient in order to obtain a duality relation of the parameters in the thermodynamic limit. This relation together with the Edwards-Sokal coupling, using  $p = 1 - e^{-\beta}$ , permits find a relation between the parameters  $(\beta, \mu)$  of the Potts model coupled to CT and the parameters  $(\beta^*, \mu^*)$  of its dual for the infinite-volume (thermodynamic limit).

The paper is organized as follows. In Section 2, we introduce notation and give a summary of the main features of CTs. We also define the annealed Potts model coupled to CTs, and we establish the main results of this work, Theorem 1 and 2. In Section 3, we describe the FK-Potts model and establish the technical Lemma 1 of duality that we will be used in next sections. Section 4 contains the proof of Theorem 1. This result will play a key role in the proof of Theorem 2. Also, in Corollary 2 we provide asymptotic behavior for the critical curve. In Section 5, utilizing the High-T expansion for  $q$ -state Potts model, we prove Theorem 2. The paper concludes with Section 6, where we compare our results with the properties obtained in [20] and [14].

## 2 Notations and main results

In this section we first introduce notation and give a summary of the causal dynamical triangulation,  $q$ -state Potts model and we define the Potts model coupled CTs. Finally, we give a short of the Edwards-Sokal coupling. We refer to [19], [27], [20], for more details. We attempt at establishing regions where the infinite-volume free energy converges, yielding results on the con-

vergence and asymptotic properties of the partition function and the Gibbs measure.

## 2.1 Two-dimensional Lorentzian models

We will work with rooted causal dynamic triangulations of the cylinder  $C_N = \mathcal{S} \times [0, N]$ ,  $N = 1, 2, \dots$ , which have  $N$  bonds (strips)  $\mathcal{S} \times [j, j+1]$ . Here  $\mathcal{S}$  stands for a unit circle (see Figure 1 for a geometric representation of a CT). Formally, a triangulation  $\mathbf{t}$  of  $C_N$  is called a *rooted causal dynamic triangulation* (CTD) if the following conditions hold:

- each triangular face of  $\mathbf{t}$  belongs to some strip  $\mathcal{S} \times [j, j+1]$ ,  $j = 1, \dots, N-1$ , and has all vertices and exactly one edge on the boundary  $(\mathcal{S} \times \{j\}) \cup (\mathcal{S} \times \{j+1\})$  of the strip  $\mathcal{S} \times [j, j+1]$ ;
- the number of edges on  $\mathcal{S} \times \{j\}$  should be finite for any  $j = 0, 1, \dots, N-1$ : let  $n^j = n^j(\mathbf{t})$  be the number of edges on  $\mathcal{S} \times \{j\}$ , then  $1 \leq n^j < \infty$  for all  $j = 0, 1, \dots, N-1$ .

and have a root face, with the anti-clockwise ordering of its vertices  $(x, y, z)$ , where  $x$  and  $y$  lie in  $\mathcal{S} \times \{0\}$ .

The CTs arise naturally when physicists attempt to define a fundamental path integral in quantum gravity. See [1] for a review of the relevant literature. For a rigorous mathematical background of the model we refer to [19]. Additional properties of CTs have been studied in [13].

A rooted CT  $\mathbf{t}$  of  $C_N$  is identified with a *compatible sequence*

$$\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1)),$$

where  $\mathbf{t}(i)$  is a triangulation of the strip  $\mathcal{S} \times [i, i+1]$ . The compatibility means that

$$n_{up}(\mathbf{t}(i+1)) = n_{do}(\mathbf{t}(i)), \quad i = 0, \dots, N-2. \quad (1)$$

Note that for any edge lying on the slice  $\mathcal{S} \times \{i\}$  belongs to exactly two triangles: one up-triangle from  $\mathbf{t}(i)$  and one down-triangle from  $\mathbf{t}(i-1)$ . This provides the following relation: the number of triangles in the triangulation  $\mathbf{t}$ , denoted by  $n(\mathbf{t})$ , is twice the total number of edges on the slices. More precisely, remind that  $n^i$  is the number of edges on slice  $\mathcal{S} \times \{i\}$ . Then, for any  $i = 0, 1, \dots, N-1$ ,

$$n(\mathbf{t}(i)) = n_{up}(\mathbf{t}(i)) + n_{do}(\mathbf{t}(i)) = n^i + n^{i+1}. \quad (2)$$

In the usual physical approach to statistical models, the computation of the partition function is the first step towards a deep understanding of the model, enabling for instance the computation of the free energy and study phase transition of the model. Following this approach, the computation of the partition function for the case of pure CTs, was first introduced and computed in [11] (see also [19] for a mathematically rigorous account).

Since we are interested in the bulk behavior of Potts model coupled to CTs, we will in the following for simplicity choose triangulations with periodical spatial boundary conditions, i.e. the strip  $\mathbf{t}(N-1)$  is compatible with  $\mathbf{t}(0)$ , and let  $\mathbb{LT}_N$  denote the set of causal triangulations on the cylinder  $C_N$  with this boundary condition, thus the partition function for rooted CTs in the cylinder  $C_N$  with periodical spatial boundary conditions and for the value of the cosmological constant  $\mu$  is given by

$$Z_N(\mu) = \sum_{\mathbf{t}} e^{-\mu n(\mathbf{t})} = \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\}. \quad (3)$$

Moreover, the periodical spatial boundary condition on the CTs permits to write the partition function  $Z_N(\mu)$  in a trace-related form

$$Z_N(\mu) = \text{tr} (U^N). \quad (4)$$

This gives rise to a transfer matrix  $U = \{u(n, n')\}_{n, n'=1, 2, \dots}$  describing the transition from one spatial strip to the next one. It is an infinite matrix with positive entries

$$u(n, n') = \binom{n + n' - 1}{n - 1} e^{-\mu(n + n')}. \quad (5)$$

Employing the  $N$ -strip partition function for pure CTs with periodical boundary condition, defined by the formula (3), we define the  $N$ -strip Gibbs probability distribution for pure CTs

$$\mathbb{Q}_{N, \mu}(\mathbf{t}) = \frac{1}{Z_N(\mu)} e^{-\mu n(\mathbf{t})}. \quad (6)$$

In 2001, the paper [19] computed the partition function of the model and proved existence of a weak limit of the measures  $\mathbb{Q}_{N, \mu}$  for  $\mu \geq \ln 2$ , using transfer-matrix formalism and tree parametrization of Lorentzian triangulations (CTs). The weak limit measure permits describe each triangulation via a positive recurrent Markov chain in the subcritical case  $\mu > \ln 2$ , and as the branching process with geometric offspring distribution with parameter

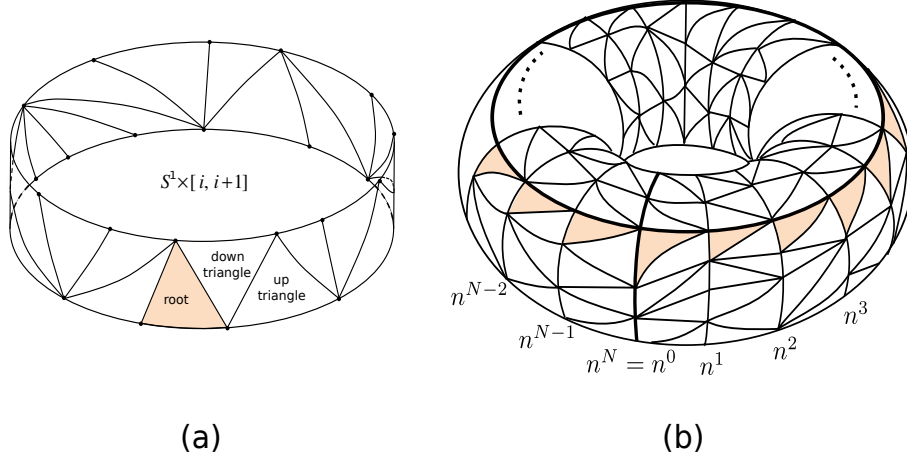


Figure 1: (a) A strip triangulation of  $\mathcal{S} \times [i, i+1]$ . (b) Geometric representation of a CT with periodic spatial boundary condition.

$1/2$ , conditioned to non-extinction at infinite in the critical case,  $\mu = \ln 2$  (see [19] for more details).

The transfer-matrix formalism suggests that, as  $N \rightarrow \infty$ , the partition function is controlled by the largest eigenvalue  $\Lambda$  of the transfer matrix (5):

$$Z_N(\mu) = \text{tr } U^N \sim \Lambda^N, \quad (7)$$

where

$$\Lambda := \Lambda(\mu) = \left[ \frac{1 - \sqrt{1 - 4 \exp(-2\mu)}}{2 \exp(-\mu)} \right]^2. \quad (8)$$

That heuristic result was proved in [20]. The following properties hold and will be utilized to prove the main results.

**Property 1.** (Theorem 1 in [19]). *For any  $\mu > \ln 2$  the following relation holds true:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N(\mu) = \ln \Lambda(\mu). \quad (9)$$

*Further, the  $N$ -strip Gibbs measure  $\mathbb{Q}_{N,\mu}$  converges weakly to a limiting measure  $\mathbb{Q}_\mu$ .*

**Property 2.** (Proposition 5, [19]). *For any  $\mu < \ln 2$ , the  $N$ -strip partition function  $Z_N(\mu)$  for pure CTs exists only if*

$$\mu > \ln \left( 2 \cos \frac{\pi}{N+1} \right). \quad (10)$$

Another proof of Property 1 can be found in [20]. In order to prove this property the authors utilize the transfer-matrix formalism and Krein-Rutman theorem. Inequality (10) in Property 2 implies that if  $\mu < \ln 2$ , then there exists  $N_0 \in \mathbb{N}$  such that  $Z_N(\mu) = \infty$  if  $N > N_0$ .

## 2.2 Potts model coupled to CT

Let  $\mathbf{t}$  be a CT on the cylinder  $C_N$  with periodic boundary condition. Each triangulation  $\mathbf{t}$  can be view as a graph  $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$  embedded on a torus. Potts spin systems are generalizations of the Ising model. Whereas in Ising systems the spins on two different values, in the  $q$ -state Potts model  $q$  distinct values, represent by the elements of the set  $\{1, \dots, q\}$ , are allowed on any vertex from the triangulation  $\mathbf{t}$ . We consider the product sample space  $\Omega(\mathbf{t}) = \{1, \dots, q\}^{V(\mathbf{t})}$  and we consider a usual (ferromagnetic)  $q$ -state Potts model energy, where two spins  $\sigma(t)$  and  $\sigma(t')$  interact if their supporting vertices  $t, t'$  are connected by an common edge; such vertices are called nearest neighbors, and this property is reflected in the notation  $\langle t, t' \rangle$ . Thus, the Hamiltonian used for the  $q$ -state Potts model on  $\mathbf{t}$  is given by

$$\mathbf{h}(\boldsymbol{\sigma}) = - \sum_{\langle t, t' \rangle} \delta_{\sigma(t), \sigma(t')}. \quad (11)$$

The partition function for the  $q$ -state Potts model on  $\mathbf{t}$  is define by

$$Z_P(\beta, q, \mathbf{t}) = \sum_{\boldsymbol{\sigma}} \exp\{-\beta \mathbf{h}(\boldsymbol{\sigma})\}, \quad (12)$$

where the summation is over any configurations  $\boldsymbol{\sigma} \in \{1, \dots, q\}^{V(\mathbf{t})}$ . Thus, the  $q$ -state Potts measure on  $\mathbf{t}$  is define as follows

$$\mu_{\beta, q}^{\mathbf{t}}(\boldsymbol{\sigma}) = \frac{1}{Z_P(\beta, q, \mathbf{t})} \exp\{-\beta \mathbf{h}(\boldsymbol{\sigma})\}. \quad (13)$$

Using the partition function for the  $q$ -state Potts model on a fixed  $\mathbf{t}$ , we define the partition function for the annealed  $q$ -state Potts model coupled to CTs, at inverse temperature  $\beta > 0$  and cosmological constant  $\mu$ , as follows

$$\Xi_N(\beta, \mu) = \sum_{\mathbf{t}} \exp\{-\mu n(\mathbf{t})\} Z_P(\beta, q, \mathbf{t}) \quad (14)$$

where  $n(\mathbf{t})$  stands for the number of triangles in the triangulation  $\mathbf{t}$ . Similarly, we introduce the  $N$ -strip Gibbs probability distribution associated with (14)

$$\mathbb{P}_N^{\beta, \mu}(\mathbf{t}, \boldsymbol{\sigma}) = \frac{1}{\Xi_N(\beta, \mu)} \exp\{-\mu n(\mathbf{t}) - \beta \mathbf{h}(\boldsymbol{\sigma})\}, \quad (15)$$

and we denote by  $\mathcal{G}_{\beta,\mu}$  the set of *Gibbs measures* given by the closed convex hull of the set of weak limits:

$$\mathbb{P}^{\beta,\mu} = \lim_{N \rightarrow \infty} \mathbb{P}_N^{\beta,\mu}, \quad (16)$$

In general, the  $q$ -state Potts model can be defined on a general lattice  $G$ . Therefore, it is possible to define the  $q$ -state Potts model on the dual  $\mathbf{t}^*$  of the triangulation  $\mathbf{t}$  (see next section for a formal definition of  $\mathbf{t}^*$ ). The partition function for the  $q$ -state Potts model on  $\mathbf{t}^*$  will denote by  $Z_P(\beta^*, q, \mathbf{t}^*)$ . Finally, we define the partition function for the  $q$ -state Potts model coupled to dual CTs  $\Xi_N^*(\beta^*, \mu^*)$  as follow

$$\Xi_N^*(\beta^*, \mu^*) = \sum_{\mathbf{t}} \exp\left\{-\mu^* n(\mathbf{t})\right\} Z_P(\beta^*, q, \mathbf{t}^*). \quad (17)$$

### 2.3 Main results

In the present article, we prove the following duality relation.

**Theorem 1.** *Let  $q \geq 2$ . The free energy of the  $q$ -state Potts model coupled to causal triangulation and its dual satisfied the following duality relation*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N^*(\beta^*, \mu^*) \quad (18)$$

where  $\Xi_N, \Xi_N^*$  denote the partition function of the  $q$ -state Potts model coupled to CT and coupled dual CT respectively (defined in Section 2.2), and

$$\beta^* = \ln \left( 1 + \frac{q}{e^\beta - 1} \right), \quad \mu^* = \mu - \frac{3}{2} \ln(e^\beta - 1) + \ln q. \quad (19)$$

Thus, equation (18) relates the free energy of the  $q$ -state Potts model coupled to CTs with the free energy of its dual, and maps the high and low temperature of the dual models onto each other. We will use the duality relation of Theorem 1 and the high-temperature expansion for the  $q$ -state Potts model for determine a region in the quadrant of parameters where the critical curve for the  $q$ -state Potts model coupled CTs and its dual can be located (see Figure 2).

Understanding by *critical curve* of the model the boundary of the domain of parameters  $\beta$  and  $\mu$  ( $\beta^*$  and  $\mu^*$  on its dual, respectively) where the model exhibits subcritical behavior, this paper makes a rigorous derivation of the subcriticality domain for an  $q$ -Potts model coupled to two-dimensional



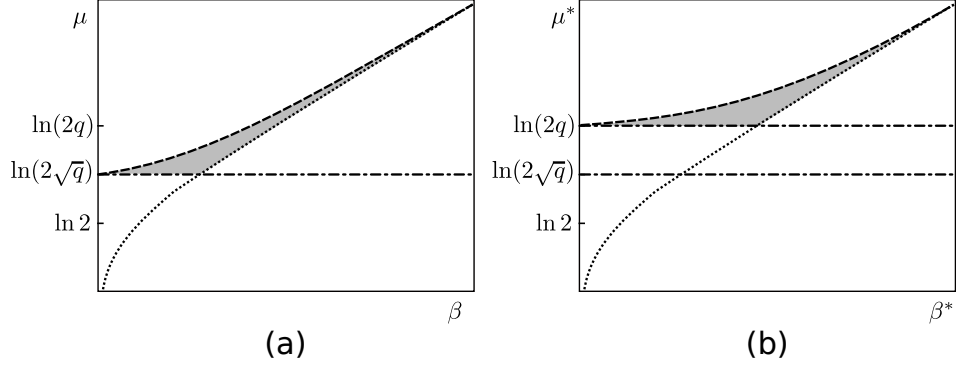


Figure 2: Illustrating the region where the critical curve for Potts model coupled CTs and its dual can be located.

CT and a domain where the typical infinite-volume Gibbs measure there no exists. The proof involve two techniques: the duality relation, Theorem 1, and high-temperature expansion for the  $q$ -state Potts model. In Figure 2, we show the region where the critical curve of the model should be located (gray region), and this figure show that critical curve for the model is asymptotic to  $\frac{3}{2}\beta + \ln 2$ .

Define the sets

$$\Sigma = \left\{ (\beta, \mu) \in \mathbb{R}_+^2 : \mu < \max \left\{ \ln(2\sqrt{q}), \frac{3}{2} \ln(e^\beta - 1) + \ln 2 \right\} \right\},$$

and

$$\Sigma^* = \left\{ (\beta^*, \mu^*) \in \mathbb{R}_+^2 : \mu^* < \max \left\{ \ln(2q), \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2 \right\} \right\}.$$

We prove the following theorem for existence and no existence of Gibbs measure for the model.

**Theorem 2.** *Let  $q \geq 2$ .*

- (a) *Potts model coupled to CTs. If  $(\beta, \mu) \in \Sigma$  then there exists  $N_0 \in \mathbb{N}$  such that the partition function  $\Xi_N(\beta, \mu) = +\infty$  whenever  $N > N_0$ . Moreover, the Gibbs distribution  $\mathbb{P}_N^{\beta, \mu}$  with periodic boundary conditions cannot be defined by using the standard formula with  $\Xi_N(\beta, \mu)$*

as a normalising denominator, consequently, there is no limiting probability measure  $\mathbb{P}^{\beta, \mu}$  as  $N \rightarrow \infty$ . Furthermore, if  $(\beta, \mu)$  satisfied

$$\mu > \frac{3}{2} \ln(q + e^\beta - 1) + \ln 2 - \ln q + \frac{3}{2} \ln \left( 1 + (q^{2/3} - 1) \frac{e^\beta - 1}{q + e^\beta - 1} \right), \quad (20)$$

the infinite-volume free energy exists, i.e. the following limit there exists:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta, \mu).$$

Moreover, as  $N \rightarrow \infty$ , the Gibbs distribution  $\mathbb{P}_N^{\beta, \mu}$  converges weakly to a limiting probability distribution  $\mathbb{P}^{\beta, \mu}$ .

- (b) Potts model coupled to dual CTs. If  $(\beta^*, \mu^*) \in \Sigma^*$  then we have the same conclusion for the the Gibbs distribution  $\mathbb{P}_N^{\beta^*, \mu^*}$ , i.e. there is no limiting probability measure  $\mathbb{P}^{\beta^*, \mu^*}$  as  $N \rightarrow \infty$ . Furthermore, if  $(\beta^*, \mu^*)$  satisfied

$$\mu^* > \frac{3}{2} \beta^* + \ln 2 + \frac{3}{2} \ln \left( 1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right), \quad (21)$$

the infinite-volume free energy exists and, as  $N \rightarrow \infty$ , the Gibbs distribution  $\mathbb{P}_N^{\beta^*, \mu^*}$  converges weakly to a limiting probability distribution  $\mathbb{P}^{\beta^*, \mu^*}$ .

As a byproduct, the Theorem 2 serves to find lower and upper bounds for the infinite-volume free energy. Moreover, in the case of 2-state Potts model (Ising model), Theorem 2 extends earlier results from [25], [20] and improves the numerical approximation of the curve in high temperature given in [14]. In addition, this approach allows to get a better approximation of the critical curve and check the asymptotic behavior of the critical curve given in [14], and it say that critical curve is asymptotic to  $\frac{3}{2}\beta + \ln 2$ . Furthermore, we show that the behavior for the matter of the free energy density in the gravitational ensemble in the thermodynamic limit, suppose in [14] for the numerical simulations, is true. In Theorem 2, we find a lower and an upper curve that converges fast to  $\frac{3}{2}\beta + \ln 2$ .

### 3 FK-Potts model on causal triangulations

In this section we describe the FK-Potts model on causal triangulations, and in the last subsection we compute inequalities which will be used in the proof of the Theorem 1.

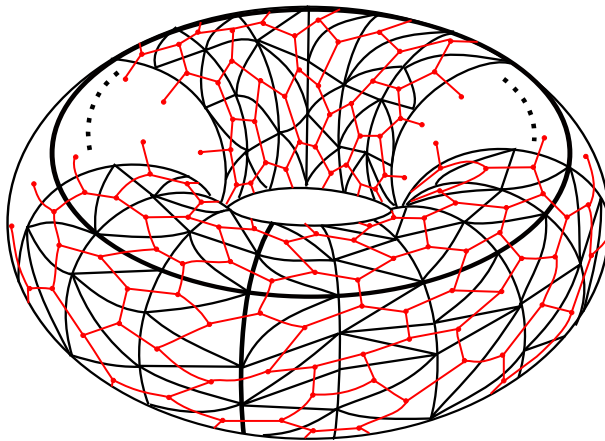


Figure 3: Geometric representation of a dual causal triangulation  $\mathbf{t}^*$  with periodic spatial boundary condition.

### 3.1 Definition of the FK-Potts model

Now, we turn to the FK representation of the  $q$ -state Potts model. The random cluster model was originally introduced by Fortuin and Kasteleyn [21] and it can be understood as an alternative representation of the  $q$ -state Potts model. This representation will be referred to as the FK representation or FK-Potts model. We are interested in study FK-Potts model on CTs and dual CTs, and find a duality relation between the parameters of the model on CTs and its dual in order to obtain information about the critical curve. In [20], [25], the model was defined putting spins on any triangle (faces), but it is equivalent to put spins on any vertex of the dual graph. In this section we work with causal triangulations and its dual with periodic boundary conditions, i.e., causal triangulations embedded in a torus  $\mathbb{T}$  (see Figure 1 (b)). In general, let  $G = (V, E)$  be a graph embedded in  $\mathbb{T}$ , we obtain its dual graph  $G^* = (V^*, E^*)$  as follows: we place a dual vertex within each face of  $G$ . For each  $e \in E$  we place a dual  $e^* = \langle x^*, y^* \rangle$  joining the two dual vertices lying in the two faces of  $G$  abutting  $e$ . Thus,  $V^*$  is in one-one correspondence with the set of faces of  $G$ , and  $E^*$  is a one-one correspondence with  $E$ . For each causal triangulation  $\mathbf{t}$ , we denote by  $\mathbf{t}^*$  its dual. Let  $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$  be a causal triangulation with periodic boundary condition, where  $V(\mathbf{t}), E(\mathbf{t})$  denote the set of vertices and edges, respectively. The state space for the FK-Potts model is the set  $\Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$ , containing configurations that allocate 0's and 1's to the edge

$e = \{i, j\} \in E(\mathbf{t})$ . For  $w \in \Sigma(\mathbf{t})$ , we call an edge  $e$  open if  $w(e) = 1$ , and closed if  $w(e) = 0$ . For  $w \in \Sigma(\mathbf{t})$ , let  $\eta(w) = \{e \in E(\mathbf{t}) : w(e) = 1\}$  denote the set of open edges. Thus, each  $w \in \Sigma(\mathbf{t})$  splits  $V(\mathbf{t})$  into the disjoint union of maximal connected components, which are called the open clusters of  $\Sigma(\mathbf{t})$ . We denote by  $k(w)$  the number of connected components (open clusters) of the graph  $(V(\mathbf{t}), \eta(w))$ , and note that  $k(w)$  includes a count of isolated vertices. Two sites of  $\mathbf{t}$  are said to be connected if one can be reached from another via a chain of open bonds. The partition function of the FK-Potts model on  $\mathbf{t}$  with parameters  $p$  and  $q$  and periodic boundary condition is defined by

$$Z_{FK}(p, q, \mathbf{t}) = \sum_{w \in \Sigma(\mathbf{t})} \left( \prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)}, \quad (22)$$

Thus, the FK-Potts measure on  $\mathbf{t}$  is define as follows

$$\Phi_{p,q}^{\mathbf{t}}(w) = \frac{1}{Z_{FK}(p, q, \mathbf{t})} \left( \prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)}. \quad (23)$$

We will use a similarly notation for the FK-Potts model on dual triangulation  $\mathbf{t}^*$ . We denote by  $Z_{FK}(p^*, q, \mathbf{t}^*)$  and  $\Phi_{p^*,q}^{\mathbf{t}^*}$  the partition function and the FK-Potts measure on  $\mathbf{t}^*$  with parameters  $p^*$  and  $q$ , respectively.

### 3.2 Edwards-Sokal coupling

There are several ways to make the connection between the Potts and FK-Potts model. The correspondence between the  $q$ -state Potts model and FK-Potts model was established by Fortuin and Kasteleyn [21] (see also [23], [27]). In a modern approach, these two models are related via a coupling, i.e., coupled the two systems on a common probability space. This coupling was introduced by Edwards-Sokal in [23].

Let  $\mathbf{t}$  be a CT on the cylinder  $C_N$  with periodic boundary condition. We consider the product sample space  $\Omega(\mathbf{t}) \times \Sigma(\mathbf{t})$  where  $\Omega(\mathbf{t}) = \{1, 2, \dots, q\}^{V(\mathbf{t})}$  and  $\Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$ . The Edwards-Sokal measure  $\mathcal{Q}$  on  $\Omega(\mathbf{t}) \times \Sigma(\mathbf{t})$  is define by

$$\mathcal{Q}(\sigma, w) \propto \prod_{e=\{i,j\} \in E(\mathbf{t})} \left\{ (1-p)\delta_{w(e),0} + p\delta_{w(e),1}\delta_{\sigma_i,\sigma_j} \right\}$$

**Theorem 3** (Edwards-Sokal [23]). *Let  $q \in \{2, 3, \dots\}$ . Let  $p \in (0, 1)$  and  $\mathbf{t}$  a CT with periodic boundary condition, and suppose that  $p = 1 - e^{-\beta}$ . If the configuration  $w$  is distributed according to an FK-Potts measure with parameters  $(p, q)$  on  $\mathbf{t}$ , then  $\sigma$  is distributed according to a  $q$ -state Potts measure with inverse temperature  $\beta$ . Furthermore, the Edwards-Sokal measure provides a coupling of  $\mu_{\beta, q}^{\mathbf{t}}$  and  $\Phi_{p, q}^{\mathbf{t}}$ , i.e.*

$$\sum_{w \in \Sigma(\mathbf{t})} \mathcal{Q}(\sigma, w) = \mu_{\beta, q}^{\mathbf{t}}(\sigma),$$

for all  $\sigma \in \Omega(\mathbf{t})$ , and

$$\sum_{\sigma \in \Omega(\mathbf{t})} \mathcal{Q}(\sigma, w) = \Phi_{p, q}^{\mathbf{t}}(w),$$

for all  $w \in \Sigma(\mathbf{t})$ . Moreover, we have the relation between partition functions

$$Z_{FK}(p, q, \mathbf{t}) = e^{-\beta|E(\mathbf{t})|} Z_P(\beta, q, \mathbf{t}). \quad (24)$$

### 3.3 FK-Potts model coupled to CTs with periodic boundary conditions

In this section we obtain a relation between the partition functions of FK-Potts model on a triangulation  $\mathbf{t}$  and its dual. This relation was studied by Beffara and Duminil-Copin for the FK-Potts model on  $\mathbb{Z}^2$  with free, wired and periodic boundary condition (see [24]). We will view with the dual of a FK-Potts model defined on a torus is an almost FK-Potts model, but it is not very different from one.

Let  $\mathbf{t}$  and  $\mathbf{t}^*$  a CT with periodic boundary condition and its dual. Each configuration  $w \in \Sigma(\mathbf{t}) = \{0, 1\}^{E(\mathbf{t})}$  gives rise to a dual configuration  $w^* \in \Sigma(\mathbf{t}^*) = \{0, 1\}^{E(\mathbf{t}^*)}$  given by  $w^*(e^*) = 1 - w(e)$ . That is,  $e^*$  is declared open if and only if the corresponding bond  $e$  is closed. The new configuration  $w^*$  is called the dual configuration of  $w$ , and note that there exists an one-one correspondence between  $\Sigma(\mathbf{t})$  and  $\Sigma(\mathbf{t}^*)$ . As in the Section 3, to each configuration  $w^*$  there corresponds the set  $\eta(w^*) = \{e^* \in E(\mathbf{t}^*) : w^*(e^*) = 1\}$  of its open edges.

Now, we start of one FK-Potts model on  $\mathbf{t}$  with parameters  $p$  and  $q$ , and we will obtain two FK-Potts models on the dual triangulation  $\mathbf{t}^*$  with parameters  $p^*$  and  $q$  that limited the initial model.

Let  $o(w)$  (resp.  $c(w)$ ) denote the number of open edges (resp. closed) of  $w$ ,  $k(w)$  the number of connected components of  $w$ , and  $f(w)$  the number

of faces delimited by  $w$ , i.e. the number of connected components of the complement of the set of open bonds. Now, We will define an additional parameters  $\delta(w)$  which is associated with the topology of the surface where the graph was embedded. Call a connected component of  $w$  a *net* if it contains two non-contractible simple loops  $\gamma_1, \gamma_2$  of different homotopy classes, and a *cycle* if it contain a non-contractible simple loops  $\gamma_1$  but is not a net. These definitions were introduced in [24]. In additional, notice that every configuration  $w$  can be of one three types:

- One of the cluster of  $w$  is a net. In that case, we let  $\delta(w) = 2$ ;
- One of the cluster of  $w$  is a cycle. We then let  $\delta(w) = 1$ ;
- None of the cluster of  $w$  is a net or a cycle. We let  $\delta(w) = 0$ .

Utilizing the previous definition of the parameter  $\delta$ , we obtained the following version of Euler's formula.

**Proposition 1** (Euler's formula). *Let  $\mathbf{t}$  a CT with periodic boundary condition and  $w \in \{0, 1\}^{E(\mathbf{t})}$ . Then*

$$|V(\mathbf{t})| - o(w) + f(w) = k(w) + 1 - \delta(w). \quad (25)$$

Employing duality and Proposition 1, we have the following relations

$$o(w) + o(w^*) = |E(\mathbf{t})|, \quad f(w) = k(w^*) \quad \text{and} \quad \delta(w) + \delta(w^*) = 2. \quad (26)$$

Let  $q \in [2, \infty)$  and  $p \in (0, 1)$ . The partition function for the FK-Potts model is given by

$$\begin{aligned} Z_{FK}(p, q, \mathbf{t}) &= \sum_{w \in \Sigma(\mathbf{t})} \left( \prod_{e \in E(\mathbf{t})} (1-p)^{1-w(e)} p^{w(e)} \right) q^{k(w)} \\ &= \sum_{w \in \Sigma(\mathbf{t})} p^{o(w)} (1-p)^{c(w)} q^{k(w)}. \end{aligned}$$

Employing Euler's formula and relations (26), we rewrite the number of cluster of  $w$  in terms of its dual  $w^*$

$$k(w) = |V(\mathbf{t})| - |E(\mathbf{t})| + o(w^*) + k(w^*) + 1 - \delta(w^*).$$

Note also that  $o(w) + o(w^*) = |E(\mathbf{t})| = |E(\mathbf{t}^*)|$ . Plugging before relations into the partition function of the FK-Potts model, we obtain

$$Z_{FK}(p, q, \mathbf{t}) = p^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| - |E(\mathbf{t})|} \sum_{w \in \Sigma(\mathbf{t})} \left( \frac{1-p}{p} \right)^{o(w^*)} q^{o(w^*) + k(w^*) + 1 - \delta(w^*)}$$

As there exists an one-one correspondence between  $\Sigma(\mathbf{t})$  and  $\Sigma(\mathbf{t}^*)$ , in the last equality we change the sum over  $\Sigma(\mathbf{t})$  by the sum over  $\Sigma(\mathbf{t}^*)$ . Thus, we obtain the following representation of the partition function in terms of configurations into  $\Sigma(\mathbf{t}^*)$

$$Z_{FK}(p, q, \mathbf{t}) = p^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| - |E(\mathbf{t})|} \sum_{w^* \in \Sigma(\mathbf{t}^*)} \left( \frac{q(1-p)}{p} \right)^{o(w^*)} q^{k(w^*) + 1 - \delta(w^*)} \quad (27)$$

Using the relation (27), we obtain the following lemma.

**Lemma 1.** *Let  $\mathbf{t}$  be a CT with periodic boundary condition. Then the following comparison inequalities both*

$$Z_{FK}(p, q, \mathbf{t}) \leq \left( \frac{p}{1-p^*} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| - |E(\mathbf{t})| + 1} Z_{FK}(p^*, q, \mathbf{t}^*) \quad (28)$$

and

$$\left( \frac{p}{1-p^*} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| - |E(\mathbf{t})| - 1} Z_{FK}(p^*, q, \mathbf{t}^*) \leq Z_{FK}(p, q, \mathbf{t}) \quad (29)$$

where  $Z_{FK}(p^*, q, \mathbf{t}^*)$  is the partition function for FK-Potts model on  $\mathbf{t}^*$  with parameters  $q$  and  $p^* = p^*(p, q)$  satisfying

$$p^*(p, q) = \frac{(1-p)q}{(1-p)q + p}, \text{ or equivalently } \frac{p^*}{1-p^*} \cdot \frac{p}{1-p} = q.$$

*Proof.* We introduce the parameter  $p^* = p^*(p, q)$  as solution of the equation

$$\frac{p^*}{1-p^*} = \frac{(1-p)q}{p}.$$

and it is plugging in equation (27). Thus, the partition function can be written in the following ways

$$Z_{FK}(p, q, \mathbf{t}) = \frac{p^{|E(\mathbf{t})|}}{(1-p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})| - |E(\mathbf{t})|} \sum_{w^* \in \Sigma(\mathbf{t}^*)} (p^*)^{o(w^*)} (1-p^*)^{c(w^*)} q^{k(w^*) + 1 - \delta(w^*)}.$$

Notice that  $-1 \leq 1 - \delta(w^*) \leq 1$ , for any  $w^* \in \Sigma(\mathbf{t}^*)$ . We denote by  $Z_{FK}(p^*, q, \mathbf{t}^*)$ , the partition function of a FK-Potts model with parameters  $p^*$  and  $q$ . Thus, we obtain the upper bound

$$Z_{FK}(p, q, \mathbf{t}) \leq \frac{p^{|E(\mathbf{t})|}}{(1 - p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})| - |E(\mathbf{t})| + 1} Z_{FK}(p^*, q, \mathbf{t}^*) ,$$

and the lower bound

$$\frac{p^{|E(\mathbf{t})|}}{(1 - p^*)^{|E(\mathbf{t}^*)|}} q^{|V(\mathbf{t})| - |E(\mathbf{t})| - 1} Z_{FK}(p^*, q, \mathbf{t}^*) \leq Z_{FK}(p, q, \mathbf{t})$$

for the partition function of FK-Potts model on  $\mathbf{t}$  with parameters  $p$  and  $q$ . Using the one-one correspondence between  $E(\mathbf{t})$  and  $E(\mathbf{t}^*)$ , we conclude the proof.  $\square$

The partition function for pure CT's has been determined as a sum over all possible triangulations of a cylinder where each configuration is weighted by a Boltzmann factor  $e^{-\mu n(\mathbf{t})}$ , where  $n(\mathbf{t})$  standing for the size of the triangulation and  $\mu$  being the cosmological constant. Thus, into two-dimensional quantum gravity the volume  $n(\mathbf{t})$  becomes an important dynamical variable for the model. Therefore, we rewrite inequalities for the partition function in Lemma 1 in terms of the dynamical variable  $n(\mathbf{t})$ . In the Table 1 we show the simple relation among  $V(\mathbf{t}), E(\mathbf{t}), V(\mathbf{t}^*), E(\mathbf{t}^*)$  and the number of triangles  $n(\mathbf{t})$  of a CT  $\mathbf{t}$ . Employing relations of Table 1, the

$\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$	$\mathbf{t}^* = (V(\mathbf{t}^*), E(\mathbf{t}^*))$
$ V(\mathbf{t})  = \frac{1}{2}n(\mathbf{t})$	$ V(\mathbf{t}^*)  = n(\mathbf{t})$
$ E(\mathbf{t})  = \frac{3}{2}n(\mathbf{t})$	$ E(\mathbf{t}^*)  = \frac{3}{2}n(\mathbf{t})$
$ \text{faces in } \mathbf{t}  = n(\mathbf{t})$	$ \text{faces in } \mathbf{t}^*  = \frac{1}{2}n(\mathbf{t})$

Table 1: Relation between the graphs  $\mathbf{t}, \mathbf{t}^*$  and  $n(\mathbf{t})$

Lemma 1 becomes be written in terms of  $n(\mathbf{t})$  as follow

**Corollary 1.** *Let  $\mathbf{t}$  be a CT with periodic boundary condition. Then the following comparison inequalities both*

$$\left(\frac{p}{1 - p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-n(\mathbf{t})} \leq \frac{Z_{FK}(p, q, \mathbf{t})}{Z_{FK}(p^*, q, \mathbf{t}^*)} \leq \left(\frac{p}{1 - p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-n(\mathbf{t})} \quad (30)$$



and

$$\left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-\frac{1}{2}n(\mathbf{t})} \leq \frac{Z_{FK}(p^*, q, \mathbf{t}^*)}{Z_{FK}(p, q, \mathbf{t})} \leq \left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-\frac{1}{2}n(\mathbf{t})} \quad (31)$$

where parameters  $q$  and  $p^* = p^*(p, q)$  satisfy

$$p^*(p, q) = \frac{(1-p)q}{(1-p)q + p}, \text{ or equivalently } \frac{p^*}{1-p^*} \frac{p}{1-p} = q.$$

## 4 Proof of Theorem 1

In the previous section we have found comparison inequalities between the partition function of the FK-Potts model on  $\mathbf{t}$  and the partition function of the FK-Potts model on its dual. In this section, we employ results of previous sections in order to prove Theorem 1. Combining inequalities (30), (31) and the Edwards-Sokal coupling, we obtain the following comparison inequalities between the partition function of the  $q$ -state Potts model on  $\mathbf{t}$  and the partition function of the  $q$ -state Potts model on its dual  $\mathbf{t}^*$ .

$$\left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-n(\mathbf{t})} e^{\frac{3}{2}(\beta-\beta^*)n(\mathbf{t})} \leq \frac{Z_P(\beta, q, \mathbf{t})}{Z_P(\beta^*, q, \mathbf{t}^*)} \leq \left(\frac{p}{1-p^*}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-n(\mathbf{t})} e^{\frac{3}{2}(\beta-\beta^*)n(\mathbf{t})} \quad (32)$$

and

$$\left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{-1-\frac{1}{2}n(\mathbf{t})} e^{\frac{3}{2}(\beta^*-\beta)n(\mathbf{t})} \leq \frac{Z_P(\beta^*, q, \mathbf{t}^*)}{Z_P(\beta, q, \mathbf{t})} \leq \left(\frac{p^*}{1-p}\right)^{\frac{3}{2}n(\mathbf{t})} q^{1-\frac{1}{2}n(\mathbf{t})} e^{\frac{3}{2}(\beta^*-\beta)n(\mathbf{t})} \quad (33)$$

where  $(e^\beta - 1)(e^{\beta^*} - 1) = q$ .

Note that inequalities (32) and (33) are satisfied for any  $\mathbf{t} \in \mathbb{LT}_N$  and for any  $N \in \mathbb{N}$ . Utilizing this observation we establish the following relation:

Let  $\mathbf{t}, \mathbf{t}^*$  be a infinite causal triangulation and its dual respectively, and denoting by  $\psi_G(\beta)$  the free energy of the Potts model defined on the graph  $G = \mathbf{t}, \mathbf{t}^*$ . Then, we obtain the following result in the thermodynamic limit.

$$\begin{aligned} \frac{1}{2}\psi_{\mathbf{t}}(\beta) - \psi_{\mathbf{t}^*}(\beta^*) &= \frac{3}{2} \ln \left( 1 - e^{-\beta} + qe^{-\beta} \right) - \ln q + \frac{3}{2}(\beta - \beta^*) \\ &= \frac{3}{2} \ln \left( e^\beta - 1 \right) - \ln q, \end{aligned} \quad (34)$$

for any  $\mathbf{t} \in \mathbb{LT}_\infty$ .

*Proof of Theorem 1.* Remember that  $p^* = 1 - e^{-\beta^*}$  and  $p = 1 - e^{-\beta}$ . Thus,

$$\frac{p^*}{1-p} = (1 - e^{-\beta^*}) + qe^{-\beta^*} = \frac{q}{(1 - e^{-\beta}) + qe^{-\beta}}$$

and

$$\frac{p}{1-p^*} = \frac{q}{(1 - e^{-\beta^*}) + qe^{-\beta^*}} = (1 - e^{-\beta}) + qe^{-\beta}.$$

Multiplying by the Boltzmann factor  $e^{-\mu n(\mathbf{t})}$  in equations (32) and (33), and sum over all possible CTs of the cylinder  $C_N$ , we obtain the following comparison inequalities between the annealed model

$$\frac{1}{q} \Xi_N^*(\beta^*, \mu^*) \leq \Xi_N(\beta, \mu) \leq q \Xi_N^*(\beta^*, \mu^*) \quad (35)$$

where  $\Xi_N$  and  $\Xi_N^*$  were defined in (14) and (17), and they denote the partition function of the  $q$ -state Potts model coupled to dual CTs and its dual, respectively, with parameters related by

$$\beta^* = \ln \left( 1 + \frac{q}{e^\beta - 1} \right), \quad \mu^* = \mu - \frac{3}{2} \ln(e^\beta - 1) + \ln q.$$

Similarly, we have

$$\frac{1}{q} \Xi_N(\beta, \mu) \leq \Xi_N^*(\beta^*, \mu^*) \leq q \Xi_N(\beta, \mu) \quad (36)$$

where

$$\beta = \ln \left( 1 + \frac{q}{e^{\beta^*} - 1} \right), \quad \mu = \mu^* - \frac{3}{2} \ln(e^{\beta^*} - 1) + \frac{1}{2} \ln q.$$

Taking the natural logarithm in inequalities (35) and (36), divide both sides of the above inequalities by  $N$ , we obtain the inequality

$$\left| \frac{\ln \Xi_N(\beta, \mu) - \ln \Xi_N^*(\beta^*, \mu^*)}{N} \right| \leq \frac{\ln q}{N}.$$

Let  $N \rightarrow \infty$  we conclude the proof of Theorem 1.  $\square$

An interesting and simple consequence of the Theorem 1 is the asymptotic behavior of the parameter associated with the random geometry in the coupled model.

**Corollary 2.** *The following asymptotic behavior for parameters  $\mu$  and  $\mu^*$  are fulfilled.*

1. If small  $\beta$  and  $\beta^*$ , we have that

$$\mu(\beta) \approx \ln 2\sqrt{q} + o(\beta^2) \quad , \quad \mu(\beta) \approx \ln 2q + o(\beta^2).$$

2. If large  $\beta$  and  $\beta^*$ , we have that

$$\mu(\beta) \approx \frac{3}{2}\beta + \ln 2 \quad , \quad \mu^*(\beta^*) \approx \frac{3}{2}\beta^* + \ln 2.$$

## 5 Proof of Theorem 2

### 5.1 Lower bound for the critical curve

Let  $\mathbf{t}$  be a CT embedded in the torus with height  $N$ . We define the set  $\Pi_i$  of configurations in  $\Sigma(\mathbf{t})$  which splits  $V(\mathbf{t})$  in  $i$  maximal connected components, i.e.

$$\Pi_i = \{w \in \Sigma(\mathbf{t}) : k(w) = i\}.$$

Similarly, we denote  $\Pi_i^*$  the set of configurations in  $\Sigma(\mathbf{t}^*)$  which splits  $V(\mathbf{t}^*)$  in  $i$  maximal connected components.

Another way of writing  $Z_P$  is as the moment generating function of the number of clusters in a random graph as follow

$$Z_P(\beta, q, G) = \exp(\beta|E(G)|) \phi_p^G \left( q^{k(w)} \right) = \exp(\beta|E(G)|) \sum_{i \geq 1} q^i \phi_p^G(\Pi_i), \quad (37)$$

where  $|E(G)|$  denotes the number of edges of the graph  $G$ , and  $\phi_p^G$  denotes product measures on  $\Sigma(G) = \{0, 1\}^{E(G)}$ .

Utilizing  $G = \mathbf{t}, \mathbf{t}^*$ , we write the representation (37) in terms of the dynamical variable  $n(\mathbf{t})$  of the model. For that, we consider the two cases of interest separately.

1. *The model on CTs:* In this case, we can to write the partition function in terms of volume  $n(\mathbf{t})$  of the triangulation as follow

$$Z_P(\beta, q, \mathbf{t}) = \exp \left( \frac{3}{2} \beta n(\mathbf{t}) \right) \phi_p^{\mathbf{t}} \left( q^{k(w)} \right). \quad (38)$$

Utilizing representation (38), we obtain two lower bounds for the partition function of the Potts model on  $\mathbf{t}$

$$Z_P(\beta, q, \mathbf{t}) \geq \max \left\{ q \left( e^\beta - 1 \right)^{\frac{3}{2}n(\mathbf{t})}, q^{\frac{1}{2}n(\mathbf{t})} \right\}. \quad (39)$$

These lower bounds for the  $q$ -state Potts model on  $\mathbf{t}$  permit to obtain a lower barrier for the parameters where the annealed model could be defined, and the partition function of the model coupled to CTs could no explode in finite volume. Apparently, the reader could think that crude bounds employed do not give thermodynamics information of the model, but these bound are sharp at high and low temperature, as we show in the proof of Theorem 2. Furthermore, these lower bounds serves in order to obtain information about the Gibbs measure for  $q$ -state Potts model coupled to CTs.

**Proposition 2.** *If  $(\beta, \mu) \in \mathbb{R}^+$  such that*

$$\mu < \max \left\{ \ln 2\sqrt{q}, \frac{3}{2} \ln(e^\beta - 1) + \ln 2 \right\},$$

*then there exists  $N_0 \in \mathbb{N}$  such that the partition function  $\Xi_N(\beta, \mu) = \infty$  whenever  $N > N_0$ . Moreover, the Gibbs distribution  $\mathbb{P}_N^{\beta, \mu}$  cannot be defined by using the standard formula with  $\Xi_N(\beta, \mu)$  as a normalising denominator, consequently, there is no limiting probability measure  $\mathbb{P}^{\beta, \mu}$  as  $N \rightarrow \infty$ .*

*Proof.* The lower bound in (39) for  $Z_P(\beta, q, \mathbf{t})$  provide the following lower bounds for the annealed partition function  $\Xi_N(\beta, \mu)$ ,

$$\Xi_N(\beta, \mu) \geq \max \left\{ qZ_N \left( \mu - \frac{3}{2} \ln(e^\beta - 1) \right), Z_N \left( \mu - \frac{1}{2} \ln q \right) \right\}. \quad (40)$$

Employing the estimation (10), we obtain which the partition function  $\Xi_N(\beta, \mu)$  there is no exist if

$$\mu \leq \frac{1}{2} \ln q + \ln \left( 2 \cos \frac{\pi}{N+1} \right) \text{ or } \mu \leq \frac{3}{2} \ln(e^\beta - 1) + \ln \left( 2 \cos \frac{\pi}{N+1} \right).$$

Letting  $N \rightarrow \infty$  we conclude the proof.  $\square$

Proposition 2 provide a region where the model cannot be defined, and the partition function of the  $q$ -state Potts model coupled to CTs is infinite. Denote by  $\Sigma$  points  $(\beta, \mu)$  in  $\mathbb{R}_+^2$  with the condition of Proposition 2. Then, by the duality relation established in Theorem 1 we obtain a region  $\Sigma^*$  where the model coupled to dual CTs also cannot be defined (see Table 2). In the next section we will prove that this lower bound is sharp at low and high temperature.

2. *The model on dual CTs:* Similarly, we write the partition function in terms of volume  $n(\mathbf{t})$  of the triangulation as follow

$$Z_P(\beta^*, q, \mathbf{t}^*) = \exp \left( \frac{3}{2} \beta^* n(\mathbf{t}) \right) \phi_p^{\mathbf{t}^*} \left( q^{k(w)} \right). \quad (41)$$

from CTs $\mathbf{t}$	$\xrightarrow{\text{by duality}}$	to dual CTs $\mathbf{t}^*$
$\mu < \frac{1}{2} \ln q + \ln 2$	$\rightarrow$	$\mu^* < \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2$
$\mu < \frac{3}{2} \ln(e^\beta - 1) + \ln 2$	$\rightarrow$	$\mu^* < \ln q + \ln 2$

Table 2: Bounds for the critical curve of the  $q$ -state Potts model on CTs generated bounds on its dual.

and utilizing this representation, we obtain the following lower bounds for the partition function of the Potts model on  $\mathbf{t}^*$

$$Z_P(\beta, q, \mathbf{t}) \geq \max \left\{ q \left( e^{\beta^*} - 1 \right)^{\frac{3}{2}n(\mathbf{t})}, q^{n(\mathbf{t})} \right\}. \quad (42)$$

In similar way as in before case, we have the following assertion about non existence of Gibbs measures for the model.

**Proposition 3.** *If  $(\beta^*, \mu^*) \in \mathbb{R}^+$  such that*

$$\mu^* < \max \left\{ \ln 2q, \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2 \right\},$$

*then there exists  $N_0 \in \mathbb{N}$  such that the partition function  $\Xi_N^*(\beta^*, \mu^*) = \infty$  whenever  $N > N_0$ . Moreover, the Gibbs distribution  $\mathbb{P}_N^{\beta^*, \mu^*}$  cannot be defined by using the standard formula with  $\Xi_N^*(\beta^*, \mu^*)$  as a normalising denominator, consequently, there is no limiting probability measure  $\mathbb{P}^{\beta^*, \mu^*}$  as  $N \rightarrow \infty$ .*

Utilizing the duality relation of Theorem 1, Proposition 3 provide a region where the Potts model coupled to CTs cannot be defined (see Table 3).

Introducing the function  $\varphi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ , given by

$$\varphi(\beta, \mu) = \left( \ln \left( 1 + \frac{q}{e^\beta - 1} \right), \mu - \frac{3}{2} \ln(e^\beta - 1) + \ln q \right), \quad (43)$$

and utilizing sets  $\Sigma$  and  $\Sigma^*$ , defined in Theorem 1, we have that  $\varphi(\Sigma) = \Sigma^*$ . Employing duality relation, in the next section we compute an upper bound for the critical curve of the Potts model on CTs and its dual. In additional, this approach allows to get behavior of the critical curve for the annealed model for low and high temperature.

from dual CTs $\mathbf{t}^*$	<u>by duality</u> $\rightarrow$	to CTs $\mathbf{t}$
$\mu^* < \ln q + \ln 2$	$\rightarrow$	$\mu < \frac{3}{2} \ln(e^\beta - 1) + \ln 2$
$\mu^* < \frac{3}{2} \ln(e^{\beta^*} - 1) + \ln 2$	$\rightarrow$	$\mu < \frac{1}{2} \ln q + \ln 2$

Table 3: Bounds for the critical curve of the  $q$ -state Potts model coupled to dual CTs generated bounds for the the critical curve of the  $q$ -state Potts model coupled to CTs.

## 5.2 Upper bound for the critical curve

In this section we utilize a High-T expansion for the Potts model introduced by Domb in [29], see also the review [31], [30].

Let  $\mathbf{t}$  be a CT with periodic boundary condition. The partition function for the Potts model on  $\mathbf{t}$  is write in the usual high-T expansion as

$$Z_P(\beta, q, \mathbf{t}) = \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{\sigma} \prod_{\langle i,j \rangle} (1 + f_{ij}) \quad (44)$$

where  $h = e^\beta - 1$  and  $f_{ij} = \frac{h}{q+h}(-1 + q\delta_{\sigma_i, \sigma_j})$ . It can be readily verified that  $\sum_{\sigma} f_{ij} = 0$  for all  $\{i, j\} \in E(\mathbf{t})$ , consequently, all subgraphs with one or more vertices of degree 1 give rise to zero contributions. Thus, the partition function can be written as follow

$$Z_P(\beta, q, \mathbf{t}) = \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{\sigma} \sum_{A \in G(\mathbf{t})} \prod_{\{i,j\} \in A} f_{ij},$$

where  $G(\mathbf{t})$  is the set of families of edges of  $\mathbf{t}$  without vertices of degree 1. Therefore, we can rewrite the partition function as

$$Z_P(\beta, q, \mathbf{t}) = \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} \sum_{A \in G(\mathbf{t})} w(A)$$

where  $w(A) = \sum_{\sigma} \prod_{\{i,j\} \in A} f_{ij}$  is a weight factor associated with the subset  $A$ .

We then proceeded to determine  $w(A)$ . An expression of  $w(A)$  for general  $A$  can be obtained by further expanding in  $w(A)$  the product  $\sum_{\sigma} \prod_{\{i,j\} \in A} f_{ij}$ .

This procedure leads to

$$w(A) = \left( \frac{h}{q+h} \right)^{|A|} \sum_{\sigma} \mathcal{P}(A)(\sigma),$$

where  $\mathcal{P}(A)(\sigma) = \prod_{e \in A} (-1 + q\delta_e(\sigma))$ , and if  $e = \{i, j\}$  then  $\delta_e(\sigma) = \delta_{\sigma_i, \sigma_j}$ . Expanding  $\mathcal{P}(A)(\sigma)$ , we have the following representation

$$\begin{aligned} \mathcal{P}(A)(\sigma) &= (-1)^{|A|} + (-1)^{|A|-1} q \sum_{e \in A} \delta_e(\sigma) + (-1)^{|A|-2} q^2 \sum_{e_1, e_2 \in A} \delta_{e_1}(\sigma) \delta_{e_2}(\sigma) \\ &\quad + \dots + (-1) q^{|A|-1} \sum_{e_1, \dots, e_{|A|-1} \in A} \delta_{e_1}(\sigma) \dots \delta_{e_{|A|-1}}(\sigma) \\ &\quad + q^{|A|} \delta_{e_1}(\sigma) \dots \delta_{e_{|A|}}(\sigma). \end{aligned}$$

We choose  $k$  edges  $\{e_1, \dots, e_k\}$  of  $A$ . These edges form a subgraph of  $A$ . Thus, we obtain

$$\sum_{\sigma} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) = q^{|V(\mathbf{t})| - k + \xi(e_1, \dots, e_k)}$$

where  $\xi(e_1, \dots, e_k)$  stands the total numbers of internal faces in each maximal connected component of  $\{e_1, \dots, e_k\}$  (number of independent circuits in  $\{e_1, \dots, e_k\}$ ). Note that this terms depends essentially on the topology of  $\{e_1, \dots, e_k\}$  (see Figure 4), but  $\xi(e_1, \dots, e_k) \leq \frac{2}{3}(k+1)$  for all  $k$ . Thus, we obtain the estimate

$$\sum_{\sigma} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) \leq q^{|V(\mathbf{t})| - k + \frac{2}{3}(k+1)} = q^{|V(\mathbf{t})| - \frac{k}{3} + \frac{2}{3}}$$

and

$$\sum_{\sigma} \sum_{e_1, \dots, e_k \in A} \delta_{e_1}(\sigma) \dots \delta_{e_k}(\sigma) \leq \binom{|A|}{k} q^{|V(\mathbf{t})| - \frac{k}{3} + \frac{2}{3}}.$$

Therefore,

$$\sum_{\sigma} \mathcal{P}(A)(\sigma) \leq q^{|V(\mathbf{t})| + \frac{2}{3}} \sum_{k=0}^{|A|} \binom{|A|}{k} (-1)^{|A|-k} (\sqrt[3]{q^2})^k = q^{|V(\mathbf{t})| + \frac{2}{3}} (\sqrt[3]{q^2} - 1)^{|A|},$$

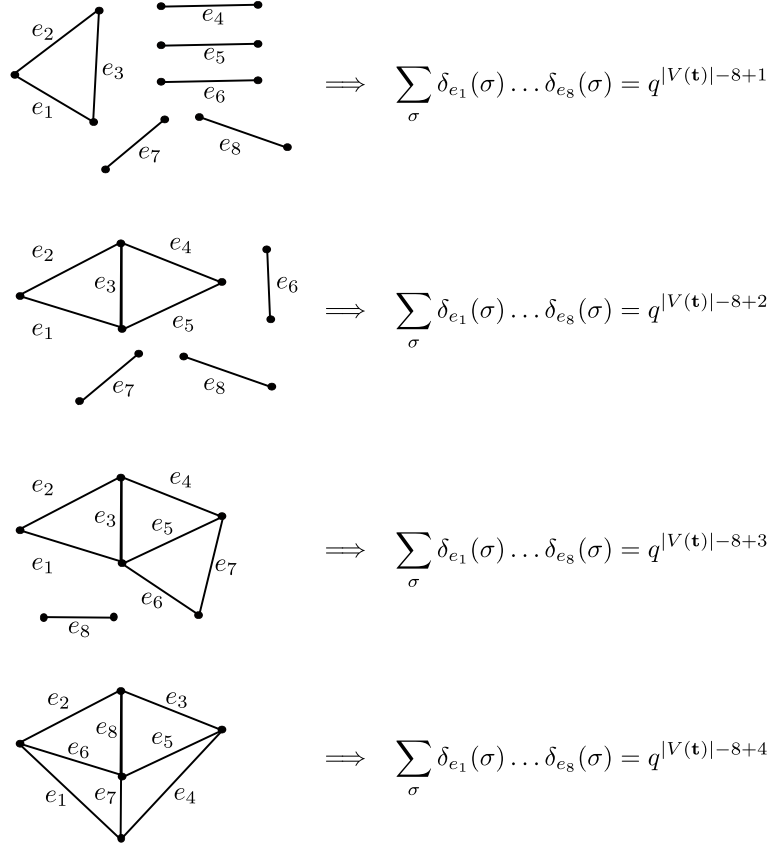


Figure 4: Examples of three subgraphs of  $A$  with 8 edges. It is clear that the term  $\xi(e_1, \dots, e_8)$  depends of the topology of the subgraphs.

and

$$\begin{aligned}
Z_P(\beta, q, \mathbf{t}) &\leq \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|+\frac{2}{3}} \sum_{A \in G(\mathbf{t})} \left( (\sqrt[3]{q^2} - 1) \frac{h}{q+h} \right)^{|A|} \\
&\leq \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})|+\frac{2}{3}} \left( 1 + \sum_{k \geq 1} \Omega_k(\mathbf{t}) u^k \right),
\end{aligned}$$

where  $\Omega_k(\mathbf{t}) = |\{A \in G(\mathbf{t}) : |A| = k\}|$  and  $u = (\sqrt[3]{q^2} - 1) \frac{h}{q+h}$ . An simple



estimation establish that  $\Omega_k(\mathbf{t}) \leq \binom{|E(\mathbf{t})|}{k}$ . Thus, we obtain the estimate

$$Z_P(\beta, q, \mathbf{t}) \leq \left( \frac{q+h}{q} \right)^{|E(\mathbf{t})|} q^{|V(\mathbf{t})| + \frac{2}{3}} (1+u)^{|E(\mathbf{t})|}. \quad (45)$$

*Proof of Theorem 2.* Employing the inequality (45) and Table 1, we write the bound (45) for the partition function of the Potts model on  $\mathbf{t}$  in terms of the number of triangles  $n(\mathbf{t})$ . This inequality is true for any graph, therefore similar computations serve for the dual model. We consider the two cases of interest separately.

1. *The model on CTs:* In this case, we can write the bound (45) for the partition function in terms of volume  $n(\mathbf{t})$  of the triangulation as follow

$$Z_P(\beta, q, \mathbf{t}) \leq \left( \frac{q+h}{q} \right)^{\frac{3}{2}n(\mathbf{t})} q^{\frac{1}{2}n(\mathbf{t}) + \frac{2}{3}} (1+u)^{\frac{3}{2}n(\mathbf{t})}. \quad (46)$$

Utilizing this estimate, we obtain a new upper bound for the partition function of the  $q$ -state Potts model coupled to CTs

$$\Xi_N(\beta, \mu) \leq q^{\frac{2}{3}} Z_N(\tilde{\mu}), \quad (47)$$

where  $\tilde{\mu} = \mu - \frac{3}{2} \ln \left( \frac{q+h}{q} \right) - \frac{1}{2} \ln q - \frac{3}{2} \ln(1+u)$  and  $Z_N(\tilde{\mu})$  is the partition function for pure CTs, defined in (2), on the cylinder  $C_N$  with periodical spatial boundary conditions and for the value of the cosmological constant  $\tilde{\mu}$ . Hence, the inequality

$$\mu > \frac{3}{2} \ln \left( q + e^\beta - 1 \right) + \ln 2 - \ln q + \frac{3}{2} \ln \left( 1 + (q^{2/3} - 1) \frac{e^\beta - 1}{q + e^\beta - 1} \right) \quad (48)$$

provides a sufficient condition for subcriticality behavior of the  $q$ -state Potts model coupled to CTs. Utilizing the High-T expansion for  $q$ -state Potts model we get to obtained a better approximation of the critical curve. Inequality (48) proof the part (a) of Theorem 2.

Now, using the duality relation of Theorem 1 and Eq. (48), we obtain a new condition for subcriticality behavior of the Potts model coupled to dual CTs

$$\mu^* > \frac{3}{2} \beta^* + \ln 2 + \frac{3}{2} \ln \left( 1 + \frac{q^{2/3} - 1}{e^{\beta^*}} \right) \quad (49)$$

Inequality (49) proof the part (b) of Theorem 2. This conclude the proof of Theorem 2 because the same approach on dual triangulations does not improve the curves obtained.  $\square$

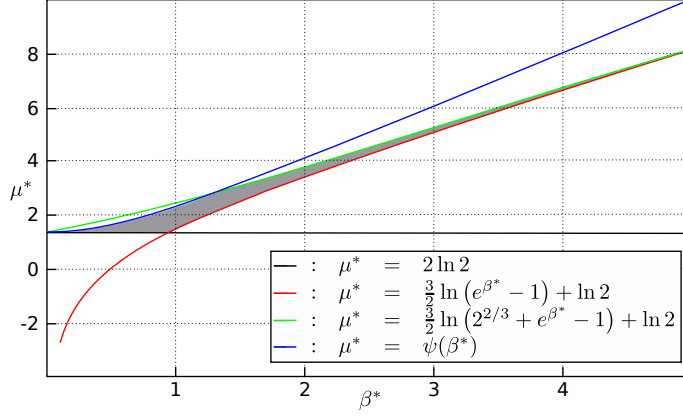


Figure 5: Region where the critical curve of the Ising model coupled to dual CDTs can be located. Curves green, black and red are establish in Theorem 2, and the curve blue was compute in [20].

## 6 $q = 2$ (Ising) system

In this section we only consider Ising model on dual causal triangulations in order to compare our results with the previous results about the Ising model coupled dual causal triangulations present in [20]. That early work utilize the transfer matrix method in order to provide a curve  $\mu^* = \psi(\beta^*)$  (blue line in Figure 5) that satisfies

$$\frac{d\psi}{d\beta^*}(0^+) = 0,$$

This property is in concordance with our result because the critical curve satisfied the same property, see Corollary 2. Now, we define the functions

$$\varphi_{inf}(\beta^*) = \max \left\{ 2 \ln 2, \frac{3}{2} \ln (e^{\beta^*} - 1) + \ln 2 \right\},$$

and

$$\varphi_{sup}(\beta^*) = \min \left\{ \psi(\beta^*), \frac{3}{2} \ln (2^{2/3} + e^{\beta^*} - 1) + \ln 2 \right\}.$$

Denoting by  $\gamma_c^I$  the critical curve of the annealed Ising model, functions  $\varphi_{inf}$  and  $\varphi_{sup}$  provide a lower and an upper bound for the critical curve of the model, that is,  $\varphi_{inf}(\beta) \leq \gamma_c^I(\beta) \leq \varphi_{sup}(\beta)$ , for any  $\beta > 0$ . The upper bound  $\varphi_{sup}$  improve bonds present in [20], [25], and confirm the expected

behavior for the critical curve described in [14]. Furthermore, in [14] authors show numerical evidence that the model have a phase transition. Denoting by  $\beta_c^I$  and  $\beta_c^{*,I}$  the critical value of the coupled Ising model and its dual, respectively, and employing the duality relation of Theorem 1, we have that

$$\beta_c^I < \frac{1}{2} \ln(1 + \sqrt{2}) < \beta_c^{*,I}. \quad (50)$$

Note that, this upper bound for the critical value of an Ising model coupled to causal triangulations do not improve the value  $\beta_c^I \approx 0.2522$  computed in [14] utilizing Monte Carlo simulation.

Finally, we have that the free energy satisfy the following inequality

$$\ln \Lambda(\mu^* - \varphi_{inf}(\beta^*) + \ln 2) < \lim_{N \rightarrow \infty} \frac{1}{N} \ln \Xi_N(\beta^*, \mu^*) < \ln \Lambda(\mu^* - \varphi_{sup}(\beta^*) + \ln 2).$$

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## References

- [1] J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum geometry. A statistical field theory approach*. No. 1 in Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, UK, 1997.
- [2] W. T. Tutte, “A census of planar triangulations,” *Can. J. Math.* **14** (1962) 21–38.
- [3] W. T. Tutte, “A census of planar maps,” *Can. J. Math.* **15** (1963) 249–271.
- [4] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, “2-d gravity and random matrices,” *Phys. Rept.* **254** (1995) 1–133, [hep-th/9306153](#).
- [5] G. Schaeffer, “Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees,” *Elec. J. Comb.* **4** (1997) R20.

- [6] J. Bouttier, P. Di Francesco, and E. Guitter, “Census of planar maps: From the one-matrix model solution to a combinatorial proof,” *Nucl. Phys.* **B645** (2002) 477, [cond-mat/0207682](#).
- [7] O. Angel and O. Schramm, “Uniform infinite planar triangulations,” *Comm. Math. Phys.* **241** (2003) 191–213, [math/0207153](#).
- [8] V. A. Kazakov, “Ising model on a dynamical planar random lattice: Exact solution,” *Phys. Lett.* **A119** (1986) 140–144.
- [9] D. V. Boulatov and V. A. Kazakov, “The Ising model on random planar lattice: The structure of phase transition and the exact critical exponents,” *Phys. Lett.* **186B** (1987) 379.
- [10] L. Onsager, “Crystal statistics. i. a two-dimensional model with an order-disorder transition,” *Phys. Rev.* **65** (1944) 117–149.
- [11] J. Ambjørn and R. Loll, “Non-perturbative Lorentzian quantum gravity, causality and topology change,” *Nucl. Phys.* **B536** (1998) 407–434, [hep-th/9805108](#).
- [12] B. Durhuus, T. Jonsson, and J. F. Wheeler, “On the spectral dimension of causal triangulations,” *J. Stat. Phys.* **139** (2010) 859–881, [0908.3643](#).
- [13] V. Sisko, A. Yambartsev, and S. Zohren, “A note on weak convergence results for uniform infinite causal triangulations,” *Markov Proc. Related Fields* (2013).
- [14] J. Ambjørn, K. N. Anagnostopoulos, and R. Loll, “A new perspective on matter coupling in 2d quantum gravity,” *Phys. Rev.* **D60** (1999) 104035, [hep-th/9904012](#).
- [15] D. Benedetti and R. Loll, “Quantum gravity and matter: Counting graphs on causal dynamical triangulations,” *Gen.Rel.Grav.* **39** (2007) 863–898, [gr-qc/0611075](#).
- [16] J. Ambjørn, K. N. Anagnostopoulos, R. Loll, and I. Pushkina, “Shaken, but not stirred - Potts model coupled to quantum gravity,” *Preprint* (2008) [0806.3506](#).
- [17] J. Ambjørn, R. Loll, W. Westra, and S. Zohren, “Putting a cap on causality violations in CDT,” *JHEP* **12** (2007) 017, [arXiv:0709.2784 \[gr-qc\]](#).

- [18] M. Krikun and A. Yambartsev, “Phase transition for the Ising model on the critical Lorentzian triangulation,” *Journal of Statistical Physics*, v. 148, p. 422-439, 2012. 0810.2182.
- [19] V. Malyshev, A. Yambartsev, and A. Zamyatin, “Two-dimensional Lorentzian models,” *Moscow Mathematical Journal* **1** (2001), no. 2, 1–18.
- [20] J. C. Hernández, A. Yambartsev, Y. Suhov, S. Zohren, *Bounds on the critical line via transfer matrix methods for an Ising model coupled to causal dynamical triangulations*. *Journal of Mathematical Physics*, v. **54**, p. 063301 (2013). 1301.1483.
- [21] C.M. Fortuin, R.W. Kasteleyn, *On the random-cluster model I. Introduction and relation to other models*. *Physica*. **57**, 536–564 (1972).
- [22] J. Ambjørn, J. Jurkiewicz, *The universe from scratch*. *Contemporary Physics* **47**, 103-117 (2006).
- [23] R. G. Edwards, A. D. Sokal, *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm*. *Phys. Rev.* (3)**38**, p. 2009-2012 (1988).
- [24] Beffara V., Duminil-Copin H.: *The self-dual point of the two-dimensional random-cluster model is critical for  $q \geq 1$* . *Probab. Theory Relat. Fields* **153**, p. 511-542 (2012).
- [25] Cerda-Hernández J.: *Critical region for an Ising model coupled to causal dynamical triangulations*. 1402.3251.
- [26] Grimmett, G. R.: *The stochastic random-cluster process and the uniqueness of random-cluster measures*. *Ann. Probab.* v. **23**(4), p. 1461–1510 (1995).
- [27] Grimmett, G. R.: *The random-cluster model*. Springer, Berlin (2006)
- [28] Napolitano, G. M. and Turova, T. *The Ising model on the random planar causal triangulation: bounds on the critical line and magnetization properties*.(2015) 1504.03828.
- [29] Domb, C. *Configurational studies of the Potts models*. *J. Phys. A* **7**, p.1335 (1974).
- [30] Wu, F. Y. *The Potts models*. *Reviews of Modern Physics*, Vol.54(1), pp.235-268 (1982).

- [31] Baxter R. J. *Exactly solved models in statistical mechanics* (1982).
- [32] J. C. Hernández, A. Yambartsev, S. Zohren, *On the critical probability of percolation on random causal triangulations*, to appear *BJPS*310.